Examples:

① Approximate $\dfrac{\sqrt{4.01}-2}{.01}$

② Approximate $\dfrac{e^{7+0.01}-e^{7}}{.01}$

③ Approximate $\dfrac{\tan\left(\frac{\pi}{4}+10^{-40}\right)-1}{10^{-40}}$

④ Find $\sqrt{9+.00001}$ approximately.

⑤ Find $(2^x)'$.

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① $\dfrac{\sqrt{4.01}-2}{.01} \approx \lim\limits_{h\to 0} \dfrac{\sqrt{4+h}-2}{h}$.

$\qquad\qquad = f'(x)$, where $f(x)=\sqrt{x}$? $x=4$.

Check: $f'(4) = \lim\limits_{h\to 0} \dfrac{f(4+h)-f(4)}{h}$

$= \lim\limits_{h\to 0} \dfrac{\sqrt{4+h}-\sqrt{4}}{h} = \lim\limits_{h\to 0} \dfrac{\sqrt{4+h}-2}{h}$

$f'(x) = \left(x^{1/2}\right)' = \frac{1}{2}x^{-1/2} \overset{x=4}{=} \dfrac{1}{2\sqrt{4}} = \boxed{\dfrac{1}{4}} \approx 0.25.$

Note: on calculator: $\dfrac{\sqrt{4.01}-2}{.01} = 0.249843\ldots$

② Approximate $\dfrac{e^{7+0.01}-e^{7}}{.01}$

$\approx$ deriv of e^x at $x=7$

$= \lim\limits_{h\to 0} \dfrac{e^{x+h}-e^{x}}{h} = \lim\limits_{h\to 0} \dfrac{e^{7+h}-e^{7}}{h}$

$(e^x)' = e^x \underset{}{\overset{x=7}{=}} \boxed{e^7}$ approximate value.

③ Approximate $\dfrac{\tan\left(\frac{\pi}{4}+10^{-40}\right)-1}{10^{-40}}$

$\approx \tan'\left(\frac{\pi}{4}\right) = \sec^2\left(\frac{\pi}{4}\right) = (\sqrt{2})^2 = \boxed{2}$

④ Find $\sqrt{9+.00001}$ approximately.

If $f(x)=\sqrt{x}$, $f'(x)=\dfrac{1}{2\sqrt{x}}$

$f'(x) = \lim\limits_{h\to 0}\dfrac{\sqrt{x+h}-\sqrt{x}}{h} \approx \dfrac{\sqrt{x+h}-\sqrt{x}}{h}$ (h small)

Let $x=9$, $h=.00001$

$\Rightarrow f'(9) \approx \dfrac{\sqrt{9+.00001}-\sqrt{9}}{.00001}$

$\dfrac{1}{2\sqrt{9}} = \dfrac{1}{6}$

$$\frac{1}{6} \approx \frac{\sqrt{9.00001} - 3}{.00001}$$

$$\frac{.00001}{6} \approx \sqrt{9.00001} - 3$$

$$3 + \underbrace{\frac{.00001}{6}}_{\frac{.16666\ldots}{100000}} \approx \sqrt{9.00001}$$

$$\boxed{3.0000016666 \approx \sqrt{9.00001}}$$

Hmk

$$\frac{2^{\overset{x+h}{3+10^{-50}}} - 2^{\overset{x}{3}}}{10^{-50}} \approx (2^x)' \text{ at } x = 3$$

$$f(x) = 2^x$$
$$x = 3$$
$$h = 10^{-50}$$

$$(2^x)' \text{ at } x=3 = 2^3 \cdot \ln(2)$$

$$(2^x)' = \left(e^{(\ln 2)x}\right)' = e^{(\ln 2)x} \cdot \ln(2)$$

$$= 2^x \cdot \ln(2).$$

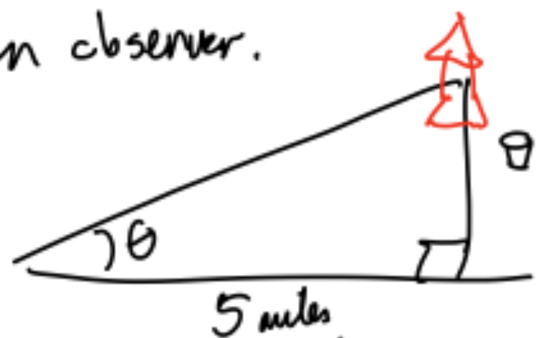
$$(a^x)' = (\ln(a)) a^x.$$

Related Rates Questions: "How fast is — changing" asking for rate

Tips:

1. Often draw picture.
2. Figure out what rate you are trying to find (eg $\frac{dy}{dt}$) & what rate(s) you are given.
3. Write down an equation that involves those variables.
4. Take derivative of both sides of the equation (use chain rule: $(\sin(y))' = \cos(y) \cdot \frac{dy}{dt} = \cos(y)\, y'$.)
5. Solve for the derivative you want.

Example Space shuttle is 5 miles from an observer.



When $\theta = 5°$

$\frac{d\theta}{dt} = 0.5°/s$

How fast is the shuttle going? $\frac{dh}{dt} = ?$

$\tan\theta = \frac{h}{5 \text{ miles}}$ $\Rightarrow$ $\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{5} \cdot \frac{dh}{dt}$

$$\left(\frac{x}{5}\right)' = \left(\frac{1}{5}\cdot x\right)' = \frac{1}{5}(x') = \frac{1}{5}.$$

Plug in: $\theta = 5^\circ \cdot \frac{\pi}{180^\circ} = \frac{\pi}{36}$, $\frac{d\theta}{dt} = \frac{0.5^\circ}{sec} = 0.5^\circ \frac{\pi}{180^\circ}/s$

$$= \frac{\pi}{360}/s.$$

$$\Rightarrow \sec^2\left(\frac{\pi}{36}\right)\cdot\left(\frac{\pi}{360}\right) = \frac{1}{5}\frac{d\theta}{dt}$$

$$\Rightarrow 5\sec^2\left(\frac{\pi}{36}\right)\left(\frac{\pi}{360}\right) = \frac{d\theta}{dt}$$

$$= 0.0439 \text{ miles}/\text{sec} =$$

$$= \frac{0.0439 \text{ miles}}{\cancel{sec}} \cdot \frac{3600 \cancel{sec}}{hour} = \boxed{158 \text{ mph}}$$